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PARTIAL FRACTION DECOMPOSITION PROOFS OF SOME q-SERIES IDENTITIES

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Abstract

Using the partial fraction decomposition method, we give new proofs of some q-series identities related to divisor functions in [10] and two finite q-series transformations in [8]. Likewise, some new q-series identities are obtained, including generalizations of some main results in [10] and generalizations of special cases of the q-Pfaff-Saalschütz summation theorem and the q-Chu-Vandermonde identity.

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1. Introduction

Wenchang Chu [2] showed that some seemingly difficult identities can be proved in a simple way by performing partial fraction decomposition to a suitable rational function, and then taking a certain limit. For further applications of the partial fraction decomposition method, see [1, 3-5].

Although for the cases that we study here, the method called q-Rice method [11] is computationally equivalent, we prefer to express everything in terms of partial fraction decomposition, which is conceptually simpler than contour integrals and residues.

We reprove in this way some identities from [10] and from [8] in this very simple fashion and obtain also some additional formulae. Especially, we generalize some main results in [10] and special cases of the q-Pfaff-Saalschütz summation theorem and the q-Chu-Vandermonde identity.

As usual, we follow the notation and terminology in [9]. For $|q| < 1$, the q-shifted factorial is defined by

$$(a;\, q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k) \quad \text{and} \quad (a;\, q)_n = \frac{(a;\, q)_\infty}{(aq^n;\, q)_\infty}, \text{ for } n \in \mathbb{Z}.$$

For convenience, we shall adopt the following notation for multiple q-shifted factorials:

$$(a_1,\, a_2,\, \ldots,\, a_m;\, q)_n = (a_1;\, q)_n (a_2;\, q)_n \cdots (a_m;\, q)_n,$$

where n is an integer or infinity.

The q-binomial coefficients, or the Gauss coefficients, are given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;\, q)_n}{(q;\, q)_k (q;\, q)_{n-k}}, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

The (unilateral) basic hypergeometric series ${}_r\phi_s$ is defined by

$$ {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(q, b_1, b_2, \ldots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k. $$

2. Some q-Series Identities of Guo-Zhang

In [10], the authors proved new generalizations of some q-series identities of Dilcher [7] and Prodinger [11] related to divisor functions. They also obtained some special cases including an identity related to overpartitions given by Corteel and Lovejoy [6, Theorem 4.4].

In this section, using the partial fraction decomposition method, we give new proofs of some theorems in [10].

Theorem 2.1 [10, Theorem 1.1]. *For* $n \geq 0$ *and* $0 \leq l, m \leq n$, *we have*

$$ \sum_{\substack{k=0 \\ k \neq m}}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q/t; q)_k (tq^{-l}; q)_{n-k}}{1 - q^{k-m}} t^k = (-1)^m q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix} (tq^{-l}; q)_l (tq^{-m}; q)_{n-l} $$

$$ \times \left(\sum_{k=0}^{n-l-1} \frac{tq^{k-m}}{1 - tq^{k-m}} - \sum_{\substack{k=0 \\ k \neq m}}^{n} \frac{q^{k-m}}{1 - q^{k-m}} \right). $$

Proof. Set

$$ F(z) = \frac{(q; q)_n}{(z; q)_{n+1}} \frac{(tz; q)_{n-l} (tq^{-l}; q)_l}{(z - q^{-m})}. $$

Performing partial fraction decomposition on $F(z)$, we have

$$ F(z) = \sum_{k=0}^{n} \frac{b_k}{1 - zq^k} + \frac{b_{n+1}}{(1 - zq^m)^2}. \tag{2.1} $$

Multiplying both sides of (2.1) by z, and then letting $z \to \infty$, we get

$$\sum_{k=0}^{n} \frac{b_k}{q^k} = 0. \tag{2.2}$$

Now we compute b_k for $k = 0, 1, \ldots, n$.

For $k \neq m$, we have

$$\begin{aligned}
b_k &= \lim_{z \to q^{-k}} (1 - zq^k) F(z) \\
&= \lim_{z \to q^{-k}} (1 - zq^k) \frac{(q;q)_n}{(z;q)_{n+1}} \frac{(tz;q)_{n-l}(tq^{-l};q)_l}{(z - q^{-m})} \\
&= \lim_{z \to q^{-k}} \frac{(q;q)_n (tz;q)_{n-l}(tq^{-l};q)_l}{(z;q)_k (zq^{k+1};q)_{n-k}(z - q^{-m})} \\
&= \frac{(q;q)_n (tq^{-k};q)_{n-l}(tq^{-l};q)_l}{(q^{-k};q)_k (q;q)_{n-k}(q^{-k} - q^{-m})} \\
&= \frac{(-1)^k q^{\binom{k+1}{2}+k} (q;q)_n (tq^{-k};q)_\infty (tq^{-l};q)_\infty}{(tq^{-k+n-l};q)_\infty (t;q)_\infty (q;q)_k (q;q)_{n-k}(1 - q^{k-m})} \\
&= \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k+1}{2}+k} (tq^{-k};q)_k (tq^{-l};q)_{n-k}}{1 - q^{k-m}} \\
&= \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q/t;q)_k (tq^{-l};q)_{n-k}}{1 - q^{k-m}} (tq)^k.
\end{aligned}$$

For $k = m$, we have

$$\begin{aligned}
b_m &= -q^{-m} \lim_{z \to q^{-m}} D_z \{(1 - zq^m)^2 F(z)\} \\
&= -q^{-m} \lim_{z \to q^{-m}} D_z \left\{ (1 - zq^m)^2 \frac{(q;q)_n}{(z;q)_{n+1}} \frac{(tz;q)_{n-l}(tq^{-l};q)_l}{(z - q^{-m})} \right\}
\end{aligned}$$

$$= \lim_{z \to q^{-m}} D_z \left\{ \frac{(q;\, q)_n (tz;\, q)_{n-l} (tq^{-l};\, q)_l}{(z;\, q)_m (zq^{m+1};\, q)_{n-m}} \right\}$$

$$= (q;\, q)_n (tq^{-l};\, q)_l \lim_{z \to q^{-m}} D_z \left\{ \frac{(tz;\, q)_{n-l}}{(z;\, q)_m (zq^{m+1};\, q)_{n-m}} \right\}$$

$$= \frac{(q;\, q)_n (tq^{-l};\, q)_l (tq^{-m};\, q)_{n-l}}{(-1)^m q^{-\binom{m+1}{2}} (q;\, q)_m (q;\, q)_{n-m}} \left(- \sum_{k=0}^{n-l-1} \frac{tq^k}{1 - tq^{k-m}} + \sum_{\substack{k=0 \\ k \neq m}}^{n} \frac{q^k}{1 - q^{k-m}} \right)$$

$$= (-1)^{m-1} q^{\binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix} (tq^{-l};\, q)_l (tq^{-m};\, q)_{n-l} \left(\sum_{k=0}^{n-l-1} \frac{tq^k}{1 - tq^{k-m}} - \sum_{\substack{k=0 \\ k \neq m}}^{n} \frac{q^k}{1 - q^{k-m}} \right).$$

Therefore, according to (2.2), we get Theorem 2.1. □

Theorem 2.2 [10, Theorem 1.2]**.** *For* $m,\ n \geq 1,$ *we have*

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q^m / t;\, q)_k (t;\, q)_{n-k}}{(tq^{-m};\, q)_{m+n} (1 - q^k)^m} t^k$$

$$= - \sum_{1 \leq k_m \leq \cdots \leq k_1 \leq n} \frac{q^{k_1}}{(1 - tq^{k_1 - 1})(1 - q^{k_1})} \cdots \frac{q^{k_m}}{(1 - tq^{k_m - m})(1 - q^{k_m})}.$$

Proof. Set

$$F(z) = \frac{(q;\, q)_n}{(z;\, q)_{n+1}} \frac{(tzq^{-m+1};\, q)_{m+n-1}}{(tq^{-m};\, q)_{m+n} (tq^{-m+1};\, q)_{m-1} (z-1)^m}.$$

Performing partial fraction decomposition on $F(z),$ we have

$$F(z) = \sum_{k=0}^{n} \frac{b_k}{1 - zq^k} + \sum_{k=2}^{m+1} \frac{c_k}{(1-z)^k}. \tag{2.3}$$

Multiplying both sides of (2.3) by z, and then letting $z \to \infty,$ we get

$$\sum_{k=0}^{n} \frac{b_k}{q^k} = 0. \tag{2.4}$$

Now we calculate b_k for $0 \le k \le n$.

For $1 \le k \le n$, we have

$$\begin{aligned}
b_k &= \lim_{z \to q^{-k}} (1 - zq^k) F(z) \\
&= \lim_{z \to q^{-k}} \frac{(q;q)_n (tzq^{-m+1};q)_{m+n-1}}{(z;q)_k (zq^{k+1};q)_{n-k} (tq^{-m};q)_{m+n} (tq^{-m+1};q)_{m-1} (z-1)^m} \\
&= \frac{(q;q)_n (tq^{-m-k+1};q)_{m+n-1}}{(q^{-k};q)_k (q;q)_{n-k} (tq^{-m};q)_{m+n} (tq^{-m+1};q)_{m-1} (q^{-k}-1)^m} \\
&= \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k+1}{2}+km} (tq^{-m-k+1};q)_{m+n-1}}{(tq^{-m};q)_{m+n} (tq^{-m+1};q)_{m-1} (1-q^k)^m} \\
&= \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k+1}{2}+km} (tq^{-m-k+1};q)_k (t;q)_{n-k}}{(tq^{-m};q)_{m+n} (1-q^k)^m} \\
&= \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q^m/t;q)_k (t;q)_{n-k}}{(tq^{-m};q)_{m+n} (1-q^k)^m} (tq)^k.
\end{aligned}$$

For $k = 0$, we have

$$\begin{aligned}
b_0 &= \frac{(-1)^m}{m!} \lim_{z \to 1} D_z^{(m)} \left\{ (1-z)^{m+1} F(z) \right\} \\
&= \frac{(-1)^m}{m!} \lim_{z \to 1} D_z^{(m)} \left\{ \frac{(-1)^m (q;q)_n (tzq^{-m+1};q)_{m+n-1}}{(zq;q)_n (tq^{-m};q)_{m+n} (tq^{-m+1};q)_{m-1}} \right\} \\
&= \frac{(q;q)_n}{(tq^{-m};q)_{m+n} (tq^{-m+1};q)_{m-1} m!} \lim_{z \to 1} D_z^{(m)} \left\{ \frac{(tzq^{-m+1};q)_{m+n-1}}{(zq;q)_n} \right\}
\end{aligned}$$

$$= \frac{(q;\, q)_n}{(tq^{-m};\, q)_m (tq^{-m+1};\, q)_{m+n-1}} [(z-1)^m] \frac{(tzq^{-m+1};\, q)_{m+n-1}}{(zq;\, q)_n}$$

$$= \frac{(q;\, q)_n}{(tq^{-m};\, q)_m (tq^{-m+1};\, q)_{m+n-1}} [w^m] \frac{(t(w+1)q^{-m+1};\, q)_{m+n-1}}{((w+1)q;\, q)_n}$$

$$= \frac{1}{(tq^{-m};\, q)_m} [w^m] \frac{\prod_{k=1}^{m+n-1} \left(1 - \frac{twq^{-m+k}}{1-tq^{-m+k}}\right)}{\prod_{k=1}^{n} \left(1 - \frac{wq^k}{1-q^k}\right)}.$$

In order to read off this coefficient, we have

$$[w^m] \frac{\prod_{k=1-m}^{n-1} \left(1 - \frac{twq^k}{1-tq^k}\right)}{\prod_{k=1}^{n} \left(1 - \frac{wq^k}{1-q^k}\right)}$$

$$= \sum_{1 \le k_m \le \cdots \le k_1 \le n} \left(\frac{q^{k_1}}{1-q^{k_1}} - \frac{tq^{k_1-1}}{1-tq^{k_1-1}}\right) \cdots \left(\frac{q^{k_m}}{1-q^{k_m}} - \frac{tq^{k_m-m}}{1-tq^{k_m-m}}\right)$$

$$= \sum_{1 \le k_m \le \cdots \le k_1 \le n} \frac{q^{k_1}(1-tq^{-1})}{(1-tq^{k_1-1})(1-q^{k_1})} \cdots \frac{q^{k_m}(1-tq^{-m})}{(1-tq^{k_m-m})(1-q^{k_m})}$$

$$= (tq^{-m};\, q)_m \sum_{1 \le k_m \le \cdots \le k_1 \le n} \frac{q^{k_1}}{(1-tq^{k_1-1})(1-q^{k_1})} \cdots \frac{q^{k_m}}{(1-tq^{k_m-m})(1-q^{k_m})}.$$

For the first equation of this chain of equalities, we argue as follows: We have to select altogether m w's from the factors. Factors in the denominator can be chosen arbitrary often, but factors from the numerator only once. Let k_1 be the largest number such that $k_1 - 1$ from the numerator is chosen or, if not, such that k_1 is chosen from the denominator. This gives a contribution

$$\left(\frac{q^{k_1}}{1-q^{k_1}}-\frac{tq^{k_1-1}}{1-tq^{k_1-1}}\right).$$

Now, let $k_2 \le k_1$ be the largest number such that k_2-2 from the numerator is chosen or, if not, such that k_2 is chosen from the denominator. This gives a contribution

$$\left(\frac{q^{k_2}}{1-q^{k_2}}-\frac{tq^{k_2-2}}{1-tq^{k_2-2}}\right).$$

Then, let $k_3 \le k_2$ be the largest number such that k_3-3 from the numerator is chosen or, if not, such that k_3 is chosen from the denominator, and so on.

Therefore, we have

$$b_0 = \sum_{1\le k_m\le\cdots\le k_1\le n} \frac{q^{k_1}}{(1-tq^{k_1-1})(1-q^{k_1})}\cdots\frac{q^{k_m}}{(1-tq^{k_m-m})(1-q^{k_m})}.$$

According to (2.4), we get Theorem 2.2. □

Theorem 2.3 [10, Theorem 4.1]**.** *For* $m, n \ge 1$, *we have*

$$\sum_{k=1}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(q^m/t;q)_k(t/q;q)_{n-k}}{(tq^{-m};q)_{m+n-1}(1-q^k)^m}t^k$$

$$= \sum_{k_1=1}^{n}\frac{q^{k_1}}{(1-tq^{k_1-2})(1-q^{k_1})}\sum_{k_2=1}^{k_1}\frac{q^{k_2}}{(1-tq^{k_2-3})(1-q^{k_2})}\cdots$$

$$\times\left(\sum_{k_m=1}^{k_{m-1}-1}\frac{tq^{k_m-m}}{1-tq^{k_m-m}}-\sum_{k_m=1}^{k_{m-1}}\frac{q^{k_m}}{1-q^{k_m}}\right).$$

Proof. Set

$$F(z) = \frac{(q;q)_n}{(z;q)_{n+1}}\frac{(tzq^{-m+1};q)_{m+n-2}}{(tq^{-m};q)_{m+n-1}(tq^{-m+1};q)_{m-2}(z-1)^m}.$$

Performing partial fraction decomposition on $F(z)$, we have

$$F(z) = \sum_{k=0}^{n} \frac{b_k}{1 - zq^k} + \sum_{k=2}^{m+1} \frac{c_k}{(1-z)^k}. \tag{2.5}$$

Multiplying by z on both sides of (2.5), and letting $z \to \infty$, we get

$$\sum_{k=0}^{n} \frac{b_k}{q^k} = 0. \tag{2.6}$$

Now we calculate b_k for $0 \le k \le n$.

For $1 \le k \le n$, we have

$$b_k = \lim_{z \to q^{-k}} (1 - zq^k) F(z)$$

$$= \lim_{z \to q^{-k}} \frac{(q;\, q)_n (tzq^{-m+1};\, q)_{m+n-2}}{(z;\, q)_k (zq^{k+1};\, q)_{n-k} (tq^{-m};\, q)_{m+n-1} (tq^{-m+1};\, q)_{m-2} (z-1)^m}$$

$$= \frac{(q;\, q)_n (tq^{-m-k+1};\, q)_{m+n-2}}{(q^{-k};\, q)_k (q;\, q)_{n-k} (tq^{-m};\, q)_{m+n-1} (tq^{-m+1};\, q)_{m-2} (q^{-k}-1)^m}$$

$$= \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k+1}{2} + km} (tq^{-m-k+1};\, q)_{m+n-2}}{(tq^{-m};\, q)_{m+n-1} (tq^{-m+1};\, q)_{m-2} (1-q^k)^m}$$

$$= \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{\binom{k+1}{2} + km} (tq^{-m-k+1};\, q)_k (t/q;\, q)_{n-k}}{(tq^{-m};\, q)_{m+n-1} (1-q^k)^m}$$

$$= \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q^m/t;\, q)_k (t/q;\, q)_{n-k}}{(tq^{-m};\, q)_{m+n-1} (1-q^k)^m} (tq)^k.$$

For $k = 0$, we have

$$b_0 = \frac{(-1)^m}{m!} \lim_{z \to 1} D_z^{(m)} \left\{ (1-z)^{m+1} F(z) \right\}$$

$$= \frac{(-1)^m}{m!} \lim_{z\to 1} D_z^{(m)} \left\{ \frac{(-1)^m (q;\, q)_n (tzq^{-m+1};\, q)_{m+n-2}}{(zq;\, q)_n (tq^{-m};\, q)_{m+n-1} (tq^{-m+1};\, q)_{m-2}} \right\}$$

$$= \frac{(q;\, q)_n}{(tq^{-m};\, q)_{m+n-1} (tq^{-m+1};\, q)_{m-2}\, m!} \lim_{z\to 1} D_z^{(m)} \left\{ \frac{(tzq^{-m+1};\, q)_{m+n-2}}{(zq;\, q)_n} \right\}$$

$$= \frac{(q;\, q)_n}{(tq^{-m};\, q)_{m+n-1} (tq^{-m+1};\, q)_{m-2}} [(z-1)^m] \frac{(tzq^{-m+1};\, q)_{m+n-2}}{(zq;\, q)_n}$$

$$= \frac{(q;\, q)_n}{(tq^{-m};\, q)_{m+n-1} (tq^{-m+1};\, q)_{m-2}} [w^m] \frac{(t(w+1)q^{-m+1};\, q)_{m+n-2}}{((w+1)q;\, q)_n}$$

$$= \frac{1}{(tq^{-m};\, q)_{m-1}} [w^m] \frac{\prod_{k=1}^{m+n-2} \left(1 - \frac{twq^{-m+k}}{1-tq^{-m+k}}\right)}{\prod_{k=1}^{n} \left(1 - \frac{wq^k}{1-q^k}\right)}.$$

In order to read off this coefficient, we have

$$[w^m] \frac{\prod_{k=1-m}^{n-2} \left(1 - \frac{twq^k}{1-tq^k}\right)}{\prod_{k=1}^{n} \left(1 - \frac{wq^k}{1-q^k}\right)}$$

$$= \sum_{k_1=1}^{n} \left(\frac{q^{k_1}}{1-q^{k_1}} - \frac{tq^{k_1-2}}{1-tq^{k_1-2}} \right) \sum_{k_2=1}^{k_1} \left(\frac{q^{k_2}}{1-q^{k_2}} - \frac{tq^{k_2-3}}{1-tq^{k_2-3}} \right) \cdots$$

$$\times \sum_{k_{m-1}=1}^{k_{m-2}} \left(\frac{q^{k_{m-1}}}{1-q^{k_{m-1}}} - \frac{tq^{k_{m-1}-m}}{1-tq^{k_{m-1}-m}} \right)$$

$$\times \left[\sum_{k_m=1}^{k_{m-1}} \left(\frac{q^{k_m}}{1-q^{k_m}} - \frac{tq^{k_m-m-1}}{1-tq^{k_m-m-1}} \right) - \left(-\frac{tq^{-m}}{1-tq^{-m}} \right) \right]$$

$$= \sum_{k_1=1}^{n} \frac{q^{k_1}(1-tq^{-2})}{(1-tq^{k_1-2})(1-q^{k_1})} \sum_{k_2=1}^{k_1} \frac{q^{k_2}(1-tq^{-3})}{(1-tq^{k_2-3})(1-q^{k_2})} \cdots$$

$$\times \sum_{k_{m-1}=1}^{k_{m-2}} \frac{q^{k_{m-1}}(1-tq^{-m})}{(1-tq^{k_{m-1}-m})(1-q^{k_{m-1}})}$$

$$\times \left(\sum_{k_m=1}^{k_{m-1}} \frac{q^{k_m}}{1-q^{k_m}} - \sum_{k_m=1}^{k_{m-1}} \frac{tq^{k_m-m-1}}{1-tq^{k_m-m-1}} + \frac{tq^{-m}}{1-tq^{-m}} \right)$$

$$= (tq^{-m};\, q)_{m-1} \sum_{k_1=1}^{n} \frac{q^{k_1}}{(1-tq^{k_1-2})(1-q^{k_1})} \sum_{k_2=1}^{k_1} \frac{q^{k_2}}{(1-tq^{k_2-3})(1-q^{k_2})} \cdots$$

$$\times \sum_{k_{m-1}=1}^{k_{m-2}} \frac{q^{k_{m-1}}}{(1-tq^{k_{m-1}-m})(1-q^{k_{m-1}})} \left(\sum_{k_m=1}^{k_{m-1}} \frac{q^{k_m}}{1-q^{k_m}} - \sum_{k_m=2}^{k_{m-1}} \frac{tq^{k_m-m-1}}{1-tq^{k_m-m-1}} \right)$$

$$= (tq^{-m};\, q)_{m-1} \sum_{k_1=1}^{n} \frac{q^{k_1}}{(1-tq^{k_1-2})(1-q^{k_1})} \sum_{k_2=1}^{k_1} \frac{q^{k_2}}{(1-tq^{k_2-3})(1-q^{k_2})} \cdots$$

$$\times \sum_{k_{m-1}=1}^{k_{m-2}} \frac{q^{k_{m-1}}}{(1-tq^{k_{m-1}-m})(1-q^{k_{m-1}})} \left(\sum_{k_m=1}^{k_{m-1}} \frac{q^{k_m}}{1-q^{k_m}} - \sum_{k_m=1}^{k_{m-1}-1} \frac{tq^{k_m-m}}{1-tq^{k_m-m}} \right).$$

The explanation is very similar to the earlier instance Theorem 2.2. The difference is that, if $k_m = 1$, there is an exception, since $k_m - m - 1 = -m$ is out of range for the numerator, and thus such a term cannot be taken.

Perhaps a more aesthetic way would be to write

$$[w^m] \frac{\prod_{k=1-m}^{n-2} \left(1 - \frac{twq^k}{1-tq^k}\right)}{\prod_{k=1}^{n} \left(1 - \frac{wq^k}{1-q^k}\right)}$$

$$= \sum_{2 \le k_m \le \cdots \le k_1 \le n} \left(\frac{q^{k_1}}{1-q^{k_1}} - \frac{tq^{k_1-2}}{1-tq^{k_1-2}} \right) \cdots \left(\frac{q^{k_m}}{1-q^{k_m}} - \frac{tq^{k_m-m-1}}{1-tq^{k_m-m-1}} \right)$$

$$+\sum_{1\le k_{m-1}\le\cdots\le k_1\le n}\left(\frac{q^{k_1}}{1-q^{k_1}}-\frac{tq^{k_1-2}}{1-tq^{k_1-2}}\right)\cdots\left(\frac{q^{k_{m-1}}}{1-q^{k_{m-1}}}-\frac{tq^{k_{m-1}-m}}{1-tq^{k_{m-1}-m}}\right)\frac{q}{1-q}$$

$$=\sum_{2\le k_m\le\cdots\le k_1\le n}\frac{q^{k_1}(1-tq^{-2})}{(1-q^{k_1})(1-tq^{k_1-2})}\cdots\frac{q^{k_m}(1-tq^{-m-1})}{(1-q^{k_m})(1-tq^{k_m-m-1})}$$

$$+\sum_{1\le k_{m-1}\le\cdots\le k_1\le n}\frac{q^{k_1}(1-tq^{-2})}{(1-q^{k_1})(1-tq^{k_1-2})}\cdots\frac{q^{k_{m-1}}(1-tq^{-m})}{(1-q^{k_{m-1}})(1-tq^{k_{m-1}-m})}\frac{q}{1-q}$$

$$=(tq^{-m-1};q)_m\sum_{2\le k_m\le\cdots\le k_1\le n}\frac{q^{k_1}}{(1-q^{k_1})(1-tq^{k_1-2})}\cdots\frac{q^{k_m}}{(1-q^{k_m})(1-tq^{k_m-m-1})}$$

$$+(tq^{-m};q)_{m-1}\sum_{1\le k_{m-1}\le\cdots\le k_1\le n}\frac{q^{k_1}}{(1-q^{k_1})(1-tq^{k_1-2})}\cdots\frac{q^{k_{m-1}}}{(1-q^{k_{m-1}})(1-tq^{k_{m-1}-m})}\frac{q}{1-q}.$$

Therefore, we have

$$b_0=\sum_{k_1=1}^{n}\frac{q^{k_1}}{(1-tq^{k_1-2})(1-q^{k_1})}\sum_{k_2=1}^{k_1}\frac{q^{k_2}}{(1-tq^{k_2-3})(1-q^{k_2})}\cdots$$
$$\times\left(\sum_{k_m=1}^{k_{m-1}}\frac{q^{k_m}}{1-q^{k_m}}-\sum_{k_m=1}^{k_{m-1}-1}\frac{tq^{k_m-m}}{1-tq^{k_m-m}}\right).$$

According to (2.6), we get Theorem 2.3. □

3. New Results

In this section, using the partial fraction decomposition method, we get new generalizations of some main results in [10], and also obtain some other new q-series identities, including generalizations of special cases of the q-Pfaff-Saalschütz summation theorem and the q-Chu-Vandermonde identity.

After that, we give proofs of three theorems given by Guo and Zhang in [10]. While we do not find a proper way to prove [10, Theorem 1.3] by using the partial fraction decomposition method, we do get the following similar new result:

Theorem 3.1. *For* $m, n \geq 1$, *we have*

$$\sum_{k=1}^{n} \frac{(q/t;q)_k (vq^m;q)_k (t;q)_{n-k} (t;q)_m}{(q;q)_k (v;q)_k (q;q)_{n-k} (q^k;q)_{m+1}} t^k$$

$$- \sum_{k=1}^{m} \frac{(tq^k;q)_{m-k} (t;q)_{n+k} (q/v;q)_k}{(q^k;q)_{n+1} (vq^{m-k};q)_k (q;q)_{m-k} (q;q)_k} v^k$$

$$= \frac{(t;q)_m (t;q)_n}{(q;q)_m (q;q)_n} \left(\sum_{k=1}^{m} \frac{q^k (1-v/q)}{(1-vq^{k-1})(1-q^k)} - \sum_{k=1}^{n} \frac{q^k (1-t/q)}{(1-tq^{k-1})(1-q^k)} \right). \quad (3.1)$$

Proof. Set

$$F(z) = -\frac{(tz;q)_n (t;q)_m (zq^{1-m}/v;q)_m}{(z;q)_{n+1} (v;q)_m (zq^{-m};q)_{m+1}} \left(\frac{v}{q}\right)^m.$$

Performing partial fraction decomposition on $F(z)$, we have

$$F(z) = \sum_{k=1}^{n} \frac{b_k}{1-zq^k} + \sum_{k=1}^{m} \frac{c_k}{1-zq^{-k}} + \frac{a_1}{1-z} + \frac{a_2}{(1-z)^2}. \quad (3.2)$$

Multiplying by z on both sides of (3.2), and letting $z \to \infty$, we get

$$\sum_{k=1}^{n} \frac{b_k}{q^k} + \sum_{k=1}^{m} \frac{c_k}{q^{-k}} = -a_1. \quad (3.3)$$

For $1 \leq k \leq n$, we have

$$b_k = \lim_{z \to q^{-k}} (1-zq^k) F(z)$$

$$= -\lim_{z \to q^{-k}} \frac{(tz;q)_n (t;q)_m (zq^{1-m}/v;q)_m}{(z;q)_k (zq^{k+1};q)_{n-k} (v;q)_m (zq^{-m};q)_{m+1}} \left(\frac{v}{q}\right)^m$$

$$= -\frac{(tq^{-k};q)_n(t;q)_m(q^{-m-k+1}/v;q)_m}{(q^{-k};q)_k(q;q)_{n-k}(v;q)_m(q^{-m-k};q)_{m+1}}\left(\frac{v}{q}\right)^m$$

$$= \frac{(-1)^k q^{\binom{k+1}{2}+k}(tq^{-k};q)_n(t;q)_m(vq^k;q)_m}{(q;q)_k(q;q)_{n-k}(v;q)_m(q^k;q)_{m+1}}$$

$$= \frac{(-1)^k q^{\binom{k+1}{2}+k}(tq^{-k};q)_k(t;q)_m(t;q)_{n-k}(vq^m;q)_k}{(q;q)_k(q;q)_{n-k}(v;q)_k(q^k;q)_{m+1}}$$

$$= \frac{(q/t;q)_k(vq^m;q)_k(t;q)_{n-k}(t;q)_m}{(q;q)_k(v;q)_k(q;q)_{n-k}(q^k;q)_{m+1}}(tq)^k.$$

For $1 \le k \le m$, we have

$$c_k = \lim_{z\to q^k}(1-zq^{-k})F(z)$$

$$= -\lim_{z\to q^k}\frac{(tz;q)_n(t;q)_m(zq^{1-m}/v;q)_m}{(z;q)_{n+1}(v;q)_m(zq^{-m};q)_{m-k}(zq^{-k+1};q)_k}\left(\frac{v}{q}\right)^m$$

$$= -\frac{(tq^k;q)_n(t;q)_m(q^{-m+k+1}/v;q)_m}{(q^k;q)_{n+1}(v;q)_m(q^{-m+k};q)_{m-k}(q;q)_k}\left(\frac{v}{q}\right)^m$$

$$= \frac{(-1)^{k+1}q^{\binom{k}{2}}(tq^k;q)_n(t;q)_m(vq^{-k};q)_m}{(q^k;q)_{n+1}(v;q)_m(q;q)_{m-k}(q;q)_k}$$

$$= \frac{(-1)^{k+1}q^{\binom{k}{2}}(tq^k;q)_{m-k}(t;q)_{n+k}(vq^{-k};q)_k}{(q^k;q)_{n+1}(vq^{m-k};q)_k(q;q)_{m-k}(q;q)_k}$$

$$= -\frac{(tq^k;q)_{m-k}(t;q)_{n+k}(q/v;q)_k}{(q^k;q)_{n+1}(vq^{m-k};q)_k(q;q)_{m-k}(q;q)_k}\left(\frac{v}{q}\right)^k.$$

Now we compute a_1

$$a_1 = -\lim_{z\to 1} D_z\{(1-z)^2 F(z)\}$$

$$= \lim_{z\to 1} D_z \left\{ \frac{(tz;\, q)_n (t;\, q)_m (zq^{1-m}/v;\, q)_m}{(zq;\, q)_n (v;\, q)_m (zq^{-m};\, q)_m} \left(\frac{v}{q}\right)^m \right\}$$

$$= \frac{(t;\, q)_m}{(v;\, q)_m} \left(\frac{v}{q}\right)^m \lim_{z\to 1} D_z \left\{ \frac{(tz;\, q)_n (zq^{1-m}/v;\, q)_m}{(zq;\, q)_n (zq^{-m};\, q)_m} \right\}$$

$$= \frac{(t;\, q)_m}{(v;\, q)_m} \left(\frac{v}{q}\right)^m \lim_{z\to 1} \left[\frac{(tz;\, q)_n (zq^{1-m}/v;\, q)_m}{(zq;\, q)_n (zq^{-m};\, q)_m} \right.$$

$$\left. \times \left(-\sum_{k=0}^{n-1} \frac{tq^k}{1-tzq^k} + \sum_{k=0}^{m-1} \frac{1}{z-vq^k} + \sum_{k=1}^{n} \frac{q^k}{1-zq^k} - \sum_{k=1}^{m} \frac{1}{z-q^k} \right) \right]$$

$$= \frac{(t;\, q)_m (t;\, q)_n (q^{1-m}/v;\, q)_m}{(v;\, q)_m (q;\, q)_n (q^{-m};\, q)_m} \left(\frac{v}{q}\right)^m$$

$$\times \left(-\sum_{k=0}^{n-1} \frac{tq^k}{1-tq^k} + \sum_{k=0}^{m-1} \frac{1}{1-vq^k} + \sum_{k=1}^{n} \frac{q^k}{1-q^k} - \sum_{k=1}^{m} \frac{1}{1-q^k} \right)$$

$$= \frac{(t;\, q)_m (t;\, q)_n}{(q;\, q)_m (q;\, q)_n} \left[\sum_{k=1}^{n} \left(\frac{q^k}{1-q^k} - \frac{tq^{k-1}}{1-tq^{k-1}} \right) + \sum_{k=1}^{m} \left(\frac{1}{1-vq^{k-1}} - \frac{1}{1-q^k} \right) \right]$$

$$= \frac{(t;\, q)_m (t;\, q)_n}{(q;\, q)_m (q;\, q)_n} \left(\sum_{k=1}^{n} \frac{q^k (1-t/q)}{(1-tq^{k-1})(1-q^k)} - \sum_{k=1}^{m} \frac{q^k (1-v/q)}{(1-vq^{k-1})(1-q^k)} \right).$$

According to (3.3), we get (3.1). □

When we set $v = t$ in (3.1), we obtain

$$\sum_{k=1}^{n} \frac{(t;\, q)_{n-k} (t;\, q)_{m+k} (q/t;\, q)_k}{(q^k;\, q)_{m+1} (q;\, q)_{n-k} (q;\, q)_k (t;\, q)_k} t^k - \sum_{k=1}^{m} \frac{(t;\, q)_{m-k} (t;\, q)_{n+k} (q/t;\, q)_k}{(q^k;\, q)_{n+1} (q;\, q)_{m-k} (q;\, q)_k (t;\, q)_k} t^k$$

$$= \frac{(1-t/q)(t;q)_m(t;q)_n}{(q;q)_m(q;q)_n}\left(\sum_{k=1}^{m}\frac{q^k}{(1-tq^{k-1})(1-q^k)} - \sum_{k=1}^{n}\frac{q^k}{(1-tq^{k-1})(1-q^k)}\right), \tag{3.4}$$

which is $v = t$ case of the following theorem given by Guo and Zhang in [10].

Theorem 3.2 [10, Theorem 1.3]**.** *For $m, n \geq 0$, we have*

$$\sum_{k=1}^{n}\frac{(q/t;q)_k(vq^m;q)_k(t;q)_{n-k}(t;q)_m}{(q;q)_k(v;q)_k(q;q)_{n-k}(q^k;q)_{m+1}}t^k - \sum_{k=1}^{m}\frac{(q/t;q)_k(vq^n;q)_k(t;q)_{m-k}(t;q)_n}{(q;q)_k(v;q)_k(q;q)_{m-k}(q^k;q)_{n+1}}t^k$$

$$= \frac{(1-t/q)(t;q)_m(t;q)_n}{(q;q)_m(q;q)_n}\left(\sum_{k=1}^{m}\frac{q^k}{(1-tq^{k-1})(1-q^k)} - \sum_{k=1}^{n}\frac{q^k}{(1-tq^{k-1})(1-q^k)}\right).$$

Interchanging m, n in (3.1), we have

$$\sum_{k=1}^{m}\frac{(q/t;q)_k(vq^n;q)_k(t;q)_{m-k}(t;q)_n}{(q;q)_k(v;q)_k(q;q)_{m-k}(q^k;q)_{n+1}}t^k$$

$$- \sum_{k=1}^{n}\frac{(tq^k;q)_{n-k}(t;q)_{m+k}(q/v;q)_k}{(q^k;q)_{m+1}(vq^{n-k};q)_k(q;q)_{n-k}(q;q)_k}v^k$$

$$= \frac{(t;q)_m(t;q)_n}{(q;q)_m(q;q)_n}\left(\sum_{k=1}^{n}\frac{q^k(1-v/q)}{(1-vq^{k-1})(1-q^k)} - \sum_{k=1}^{m}\frac{q^k(1-t/q)}{(1-tq^{k-1})(1-q^k)}\right). \tag{3.5}$$

Combining (3.1) and (3.5), we get

$$\sum_{k=1}^{n}\frac{(q/t;q)_k(vq^m;q)_k(t;q)_{n-k}(t;q)_m}{(q;q)_k(v;q)_k(q;q)_{n-k}(q^k;q)_{m+1}}t^k$$

$$- \sum_{k=1}^{m}\frac{(q/t;q)_k(vq^n;q)_k(t;q)_{m-k}(t;q)_n}{(q;q)_k(v;q)_k(q;q)_{m-k}(q^k;q)_{n+1}}t^k$$

$$= \sum_{k=1}^{m} \frac{(tq^k;\, q)_{m-k}(t;\, q)_{n+k}(q/v;\, q)_k}{(q^k;\, q)_{n+1}(vq^{m-k};\, q)_k(q;\, q)_{m-k}(q;\, q)_k} v^k$$

$$- \sum_{k=1}^{n} \frac{(tq^k;\, q)_{n-k}(t;\, q)_{m+k}(q/v;\, q)_k}{(q^k;\, q)_{m+1}(vq^{n-k};\, q)_k(q;\, q)_{n-k}(q;\, q)_k} v^k$$

$$+ \frac{(1-v/q)(t;\, q)_m(t;\, q)_n}{(q;\, q)_m(q;\, q)_n} \left(\sum_{k=1}^{m} \frac{q^k}{(1-vq^{k-1})(1-q^k)} - \sum_{k=1}^{n} \frac{q^k}{(1-vq^{k-1})(1-q^k)} \right)$$

$$+ \frac{(1-t/q)(t;\, q)_m(t;\, q)_n}{(q;\, q)_m(q;\, q)_n} \left(\sum_{k=1}^{m} \frac{q^k}{(1-tq^{k-1})(1-q^k)} - \sum_{k=1}^{n} \frac{q^k}{(1-tq^{k-1})(1-q^k)} \right).$$

According to the above identity and Theorem 3.2, we have the following identity:

$$\sum_{k=1}^{n} \frac{(tq^k;\, q)_{n-k}(t;\, q)_{m+k}(q/v;\, q)_k}{(q^k;\, q)_{m+1}(vq^{n-k};\, q)_k(q;\, q)_{n-k}(q;\, q)_k} v^k$$

$$- \sum_{k=1}^{m} \frac{(tq^k;\, q)_{m-k}(t;\, q)_{n+k}(q/v;\, q)_k}{(q^k;\, q)_{n+1}(vq^{m-k};\, q)_k(q;\, q)_{m-k}(q;\, q)_k} v^k$$

$$= \frac{(1-v/q)(t;\, q)_m(t;\, q)_n}{(q;\, q)_m(q;\, q)_n} \left(\sum_{k=1}^{m} \frac{q^k}{(1-vq^{k-1})(1-q^k)} - \sum_{k=1}^{n} \frac{q^k}{(1-vq^{k-1})(1-q^k)} \right),$$

which can be obtained by interchanging v and t in Theorem 3.2.

Remark 3.3. We were not successful to prove [10, Theorem 1.3] with the partial fraction decomposition method. However, we offer the following observation. The right hand-side of it does not depend on the parameter v. If this fact could be shown by simple means, then we could argue that the left hand-side also does not depend on v. Henceforth, we could set $v = t$, and would achieve a proof.

Next, making some changes on the rational function $F(z)$ in the proof of Theorem 2.1, we can get some new results.

Set

$$F(z) = \frac{(vq;\, q)_n (tz;\, q)_{n-l} (tq^{-l} / v;\, q)_l}{(zv;\, q)_{n+1} (z - q^{-m})}.$$

Then using the partial fraction decomposition method, we get

$$\sum_{k=0}^{n} \frac{(vq / t;\, q)_k (tq^{-l} / v;\, q)_{n-k}}{(q;\, q)_k (q;\, q)_{n-k} (1 - vq^{k-m})} \left(\frac{t}{v}\right)^k = \frac{(tq^{-m};\, q)_{n-l} (tq^{-l} / v;\, q)_l}{(vq^{-m};\, q)_{n+1}}. \tag{3.6}$$

Setting v and t to be aq^m and aq^{m+1} / b in (3.6), respectively, we get the following result:

Theorem 3.4. *For* $n \geq 0$ *and* $0 \leq l \leq n$, *we have*

$$\sum_{k=0}^{n} \frac{(a,\, b,\, q^{-n};\, q)_k}{(q,\, aq,\, bq^{-n+l};\, q)_k} (q^{l+1})^k = \frac{(q;\, q)_n (aq / b;\, q)_{n-l}}{(aq;\, q)_n (q / b;\, q)_{n-l}}. \tag{3.7}$$

Setting $l = 0$ in (3.7), we obtain

$$\sum_{k=0}^{n} \frac{(a,\, b,\, q^{-n};\, q)_k}{(q,\, aq,\, bq^{-n};\, q)_k} q^k = \frac{(q,\, aq / b;\, q)_n}{(aq,\, q / b;\, q)_n},$$

which is $c = aq$ case of the q-Pfaff-Saalschütz summation theorem [9, Appendix II.12]

$${}_3\phi_2 \left[\begin{matrix} q^{-n}, & a, & b \\ & c, & abq^{1-n} / c \end{matrix} ;\, q,\, q \right] = \frac{(c / a,\, c / b;\, q)_n}{(c,\, c / ab;\, q)_n}.$$

If we set $a = q$ in (3.7), we get

Corollary 3.5. *For* $n \geq 0$ *and* $0 \leq l \leq n$, *we have*

$$\sum_{k=0}^{n} \frac{(b,\, q^{-n};\, q)_k}{(q^2,\, bq^{-n+l};\, q)_k} (q^{l+1})^k = \frac{(q;\, q)_n (q^2 / b;\, q)_{n-l}}{(q^2;\, q)_n (q / b;\, q)_{n-l}}. \tag{3.8}$$

When we set $b = q^{n-l+1}$ in (3.8), we get

$$\sum_{k=0}^{n} \frac{(q^{n-l+1}, q^{-n}; q)_k}{(q, q^2; q)_k} (q^{l+1})^k = 0,$$

which is $a = q^{n-l+1}$ and $c = q^2$ case of the q-Chu-Vandermonde identity [9, Appendix II.7]

$${}_2\phi_1\left[\begin{matrix} q^{-n}, & a \\ & c \end{matrix}; q, \frac{cq^n}{a}\right] = \frac{(c/a; q)_n}{(c; q)_n}.$$

Moreover, we find that if we put a new parameter v in the proofs of Theorems 2.2 and 2.3, we get generalizations of these two theorems.

First, we give a generalization of Theorem 2.2.

Theorem 3.6. *For* $m, n \geq 1$, *we have*

$$\sum_{k=1}^{n} \frac{(vq; q)_n (vq^{m+1}/t; q)_{k-1} (t/v; q)_{n-k}}{(tq^{-m+1}; q)_{m+n-1} (q; q)_{k-1} (q; q)_{n-k} (1 - vq^k)^{m+1}} \left(\frac{t}{v}\right)^{k-1}$$

$$= \sum_{1 \leq k_m \leq \cdots \leq k_1 \leq n} \frac{q^{k_1 - 1}}{(1 - tq^{k_1 - 1})(1 - vq^{k_1})} \cdots \frac{q^{k_m - 1}}{(1 - tq^{k_m - m})(1 - vq^{k_m})}.$$

Proof. Set

$$F(z) = \frac{(vq; q)_n (tzq^{-m+1}; q)_{m+n-1}}{(vzq; q)_n (tq^{-m}/v; q)_m (tq^{-m+1}; q)_{m+n-1} (z-1)^{m+1}}.$$

Performing partial fraction decomposition on $F(z)$, we have

$$F(z) = \sum_{k=1}^{n} \frac{b_k}{1 - vzq^k} + \sum_{k=1}^{m+1} \frac{c_k}{(1-z)^k}. \tag{3.9}$$

Multiplying both sides of (3.9) by z, and then letting $z \to \infty$, we get

$$\sum_{k=1}^{n} \frac{b_k}{vq^k} = -c_1. \tag{3.10}$$

Now we calculate b_k for $1 \le k \le n$.

For $1 \le k \le n$, we have

$$b_k = \lim_{z \to q^{-k}/v} (1 - vzq^k) F(z)$$

$$= \lim_{z \to q^{-k}/v} \frac{(vq;\, q)_n (tzq^{-m+1};\, q)_{m+n-1}}{(vzq;\, q)_{k-1} (vzq^{k+1};\, q)_{n-k} (tq^{-m}/v;\, q)_m (tq^{-m+1};\, q)_{m+n-1} (z-1)^{m+1}}$$

$$= \frac{(vq;\, q)_n (tq^{-m-k+1}/v;\, q)_{m+n-1}}{(q^{1-k};\, q)_{k-1} (q;\, q)_{n-k} (tq^{-m}/v;\, q)_m (tq^{-m+1};\, q)_{m+n-1} (q^{-k}/v - 1)^{m+1}}$$

$$= \frac{(-1)^{k-1} q^{\binom{k+1}{2} + km} v^{m+1} (vq;\, q)_n (tq^{-m-k+1}/v;\, q)_{m+n-1}}{(q;\, q)_{k-1} (q;\, q)_{n-k} (tq^{-m}/v;\, q)_m (tq^{-m+1};\, q)_{m+n-1} (1 - vq^k)^{m+1}}$$

$$= \frac{(-1)^{k-1} q^{\binom{k+1}{2} + km} v^{m+1} (vq;\, q)_n (tq^{-m-k+1}/v;\, q)_{k-1} (t/v;\, q)_{n-k}}{(q;\, q)_{k-1} (q;\, q)_{n-k} (tq^{-m+1};\, q)_{m+n-1} (1 - vq^k)^{m+1}}$$

$$= \frac{(vq;\, q)_n (vq^{m+1}/t;\, q)_{k-1} (t/v;\, q)_{n-k}}{(q;\, q)_{k-1} (q;\, q)_{n-k} (tq^{-m+1};\, q)_{m+n-1} (1 - vq^k)^{m+1}} \left(\frac{t}{v}\right)^{k-1} v^{m+1} q^{m+k}.$$

For c_1, we have

$$c_1 = \frac{(-1)^m}{m!} \lim_{z \to 1} D_z^{(m)} \{(1-z)^{m+1} F(z)\}$$

$$= \frac{(-1)^m}{m!} \lim_{z \to 1} D_z^{(m)} \left\{ \frac{(-1)^{m+1} (vq;\, q)_n (tzq^{-m+1};\, q)_{m+n-1}}{(vzq;\, q)_n (tq^{-m}/v;\, q)_m (tq^{-m+1};\, q)_{m+n-1}} \right\}$$

$$= -\frac{(vq;\, q)_n}{(tq^{-m}/v;\, q)_m (tq^{-m+1};\, q)_{m+n-1} m!} \lim_{z \to 1} D_z^{(m)} \left\{ \frac{(tzq^{-m+1};\, q)_{m+n-1}}{(vzq;\, q)_n} \right\}$$

$$= -\frac{(vq;\, q)_n}{(tq^{-m}/v;\, q)_m (tq^{-m+1};\, q)_{m+n-1}} [(z-1)^m] \frac{(tzq^{-m+1};\, q)_{m+n-1}}{(vzq;\, q)_n}$$

$$= -\frac{(vq;\, q)_n}{(tq^{-m}/v;\, q)_m (tq^{-m+1};\, q)_{m+n-1}} [w^m] \frac{(t(w+1)q^{-m+1};\, q)_{m+n-1}}{(v(w+1)q;\, q)_n}$$

$$= -\frac{1}{(tq^{-m}/v;\, q)_m} [w^m] \frac{\prod_{k=1}^{m+n-1}\left(1 - \frac{twq^{-m+k}}{1-tq^{-m+k}}\right)}{\prod_{k=1}^{n}\left(1 - \frac{vwq^{k}}{1-vq^{k}}\right)}$$

$$= -\frac{1}{(tq^{-m}/v;\, q)_m} \sum_{1 \le k_m \le \cdots \le k_1 \le n} \left(\frac{vq^{k_1}}{1-vq^{k_1}} - \frac{tq^{k_1-1}}{1-tq^{k_1-1}}\right)$$

$$\cdots \left(\frac{vq^{k_m}}{1-vq^{k_m}} - \frac{tq^{k_m-m}}{1-tq^{k_m-m}}\right)$$

$$= -\frac{1}{(tq^{-m}/v;\, q)_m} \sum_{1 \le k_m \le \cdots \le k_1 \le n} \frac{vq^{k_1}(1-tq^{-1}/v)}{(1-tq^{k_1-1})(1-vq^{k_1})}$$

$$\cdots \frac{vq^{k_m}(1-tq^{-m}/v)}{(1-tq^{k_m-m})(1-vq^{k_m})}$$

$$= -v^m \sum_{1 \le k_m \le \cdots \le k_1 \le n} \frac{q^{k_1}}{(1-tq^{k_1-1})(1-vq^{k_1})} \cdots \frac{q^{k_m}}{(1-tq^{k_m-m})(1-vq^{k_m})}.$$

We read off the coefficient of w^m as we did in the proof of Theorem 2.2.

According to (3.10), we get Theorem 3.6. □

Setting $v = 1$ in Theorem 3.6, we get Theorem 2.2.

Setting $v = 0$ in Theorem 3.6, we have

$$\sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} (-1)^k q^{\binom{k+1}{2} - nk} = 0,$$

which is a special case of the q-binomial theorem [9, Appendix II.4]

$$ {}_1\phi_0\left[\begin{matrix} q^{-n} \\ \\ - \end{matrix}; q, z\right] = (zq^{-n}; q)_n. $$

Setting $v = q$ in Theorem 3.6, we have the following result:

Corollary 3.7. *For* $m, n \geq 1$, *we have*

$$ \sum_{k=1}^{n} \frac{(q^2; q)_n (q^{m+2}/t; q)_{k-1} (t/q; q)_{n-k}}{(tq^{-m+1}; q)_{m+n-1} (q; q)_{k-1} (q; q)_{n-k} (1-q^{k+1})^{m+1}} \left(\frac{t}{q}\right)^{k-1} $$

$$ = \sum_{1 \leq k_m \leq \cdots \leq k_1 \leq n} \frac{q^{k_1-1}}{(1-tq^{k_1-1})(1-q^{k_1+1})} \cdots \frac{q^{k_m-1}}{(1-tq^{k_m-m})(1-q^{k_m+1})}. $$

Letting $n \to \infty$ in Theorem 3.6, we have

Corollary 3.8. *For* $m \geq 1$, *we have*

$$ \sum_{k=1}^{\infty} \frac{(vq^{m+1}/t; q)_{k-1}}{(tq^{-m+1}; q)_{m-1} (q; q)_{k-1} (1-vq^k)^{m+1}} \left(\frac{t}{v}\right)^{k-1} $$

$$ = \frac{(q; q)_\infty (t; q)_\infty}{(vq; q)_\infty (t/v; q)_\infty} \sum_{1 \leq k_m \leq \cdots \leq k_1 < \infty} \frac{q^{k_1-1}}{(1-tq^{k_1-1})(1-vq^{k_1})} $$

$$ \cdots \frac{q^{k_m-1}}{(1-tq^{k_m-m})(1-vq^{k_m})}. $$

Setting $m = 1$ in Theorem 3.6, we get the following generalization of [10, Corollary 3.3].

Corollary 3.9. *For* $n \geq 1$, *we have*

$$ \sum_{k=1}^{n} \frac{(vq; q)_n (vq^2/t; q)_{k-1} (t/v; q)_{n-k}}{(t, q)_n (q; q)_{k-1} (q; q)_{n-k} (1-vq^{k_m})} \left(\frac{t}{v}\right)^{k-1} = \sum_{k=1}^{n} \frac{q^{k-1}}{(1-tq^{k-1})(1-vq^k)}. $$

Now we give the following theorem, which is a generalization of Theorem 2.3.

Theorem 3.10. *For* $m, n \geq 1$, *we have*

$$\sum_{k=1}^{n} \frac{(vq;\, q)_n (vq^{m+1}/t;\, q)_{k-1} (tq^{-1}/v;\, q)_{n-k}}{(tq^{-m+1};\, q)_{m+n-2} (q;\, q)_{k-1} (q;\, q)_{n-k} (1 - vq^k)^{m+1}} \left(\frac{t}{v}\right)^{k-1}$$

$$= \sum_{k_1=1}^{n} \frac{q^{k_1-1}}{(1 - tq^{k_1-2})(1 - vq^{k_1})} \cdots \sum_{k_{m-1}=1}^{k_{m-2}} \frac{q^{k_{m-1}-1}}{(1 - tq^{k_{m-1}-m})(1 - vq^{k_{m-1}})}$$

$$\times \left(\sum_{k_m=1}^{k_{m-1}} \frac{q^{k_m-1}}{1 - vq^{k_m}} \sum_{k_m=1}^{k_{m-1}-1} \frac{tq^{k_m-m-1}}{v - tvq^{k_m-m}} \right).$$

We perform partial fraction decomposition on

$$F(z) = \frac{(vq;\, q)_n (tzq^{-m+1};\, q)_{m+n-2}}{(vzq;\, q)_n (tq^{-m}/v;\, q)_{m-1} (tq^{-m+1};\, q)_{m+n-2} (z-1)^{m+1}}.$$

Similarly to the proof of Theorem 2.3, we can prove the above theorem. Here we omit the proof.

When we set $v = 1$ in Theorem 3.10, we get Theorem 2.3.

Setting $v = q$ in Theorem 3.10, we have

Corollary 3.11. *For* $m, n \geq 1$, *we have*

$$\sum_{k=1}^{n} \frac{(q^2;\, q)_n (q^{m+2}/t;\, q)_{k-1} (tq^{-2};\, q)_{n-k}}{(tq^{-m+1};\, q)_{m+n-2} (q;\, q)_{k-1} (q;\, q)_{n-k} (1 - q^{k+1})^{m+1}} \left(\frac{t}{q}\right)^{k-1}$$

$$= \sum_{k_1=1}^{n} \frac{q^{k_1-1}}{(1 - tq^{k_1-2})(1 - q^{k_1+1})} \cdots \sum_{k_{m-1}=1}^{k_{m-2}} \frac{q^{k_{m-1}-1}}{(1 - tq^{k_{m-1}-m})(1 - q^{k_{m-1}+1})}$$

$$\times \left(\sum_{k_m=1}^{k_{m-1}} \frac{q^{k_m-1}}{1 - q^{k_m+1}} - \sum_{k_m=1}^{k_{m-1}-1} \frac{tq^{k_m-m-2}}{1 - tq^{k_m-m}} \right).$$

Letting $n \to \infty$ in Theorem 3.10, we obtain

Corollary 3.12. *For* $m \geq 1$, *we have*

$$\sum_{k=1}^{\infty} \frac{(vq^{m+1}/t;\, q)_{k-1}}{(tq^{-m+1};\, q)_{m-1}(q;\, q)_{k-1}(1-vq^{k})^{m+1}} \left(\frac{t}{v}\right)^{k-1} = \frac{(q;\, q)_{\infty}(t;\, q)_{\infty}}{(vq;\, q)_{\infty}(tq^{-1}/v;\, q)_{\infty}}$$

$$\sum_{k_1=1}^{\infty} \frac{q^{k_1-1}}{(1-tq^{k_1-2})(1-vq^{k_1})} \cdots \sum_{k_{m-1}=1}^{k_{m-2}} \frac{q^{k_{m-1}-1}}{(1-tq^{k_{m-1}-m})(1-vq^{k_{m-1}})}$$

$$\times \left(\sum_{k_m=1}^{k_{m-1}} \frac{q^{k_m-1}}{1-vq^{k_m}} \sum_{k_m=1}^{k_{m-1}-1} \frac{tq^{k_m-m-1}}{v-tvq^{k_m-m}} \right).$$

Setting $m = 1$ in Theorem 3.10, we have the following generalization of the $l = 1$ case of [10, Corollary 3.2].

Corollary 3.13. *For* $n \geq 1$, *we have*

$$\sum_{k=1}^{n} \frac{(vq;\, q)_n (vq^2/t;\, q)_{k-1}(tq^{-1}/v;\, q)_{n-k}}{(t,\, q)_{n-1}(q;\, q)_{k-1}(q;\, q)_{n-k}(1-vq^{k})^2} \left(\frac{t}{v}\right)^{k-1} = \sum_{k=1}^{n} \frac{q^{k-1}}{1-vq^{k}} - \sum_{k=1}^{n-1} \frac{tq^{k-2}}{v-tvq^{k-1}}.$$

4. Two Identities of Fang

We notice that by using partial fraction decomposition technique, we can also give new proofs of two identities found by Fang in [8].

Theorem 4.1 [8, Corollary 3.4]**.** *We have*

$$\sum_{j=0}^{M} \begin{bmatrix} M \\ j \end{bmatrix} (-1)^j q^{\binom{j}{2}+2j} \frac{1}{1-zq^{j}} = \frac{(q;\, q)_M}{(z;\, q)_{M+1}} \sum_{j=0}^{M} (z;\, q)_j q^j.$$

Proof. Set

$$F(z) = \frac{(q;\, q)_M}{(z;\, q)_{M+1}} \sum_{j=0}^{M} (z;\, q)_j q^j.$$

Performing partial fraction decomposition on $F(z)$, we have

$$F(z) = \sum_{k=0}^{M} \frac{b_k}{1-zq^{k}}. \tag{4.1}$$

For $0 \le k \le M$, we have

$$\begin{aligned} b_k &= \lim_{z \to q^{-k}} (1 - zq^k) F(z) \\ &= \lim_{z \to q^{-k}} \frac{(q;\, q)_M}{(z;\, q)_k (zq^{k+1};\, q)_{M-k}} \sum_{j=0}^{M} (z;\, q)_j q^j \\ &= \frac{(q;\, q)_M}{(q^{-k};\, q)_k (q;\, q)_{M-k}} \sum_{j=0}^{M} (q^{-k};\, q)_j q^j \\ &= (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} M \\ k \end{bmatrix} \sum_{j=0}^{k} (q^{-k};\, q)_j q^j \\ &= \begin{bmatrix} M \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}+2k}. \end{aligned}$$

The last equation of this chain of equalities follows from the q-Chu-Vandermonde identity [9, Appendix II.6]

$$ {}_2\phi_1 \left[\begin{matrix} q^{-n}, & a \\ & c \end{matrix} ;\, q,\, q \right] = \frac{(c/a;\, q)_n}{(c;\, q)_n} a^n, \tag{4.2} $$

by setting $a = q$ and $c = 0$.

According to (4.1), we get Theorem 4.1. □

Theorem 4.2 [8, Corollary 4.1]**.** *We have*

$$\sum_{j=0}^{M} \frac{q^j}{(z;\, q)_{j+1}} = \sum_{j=0}^{M} \frac{(-1)^j q^{\binom{j}{2}+2j}}{(1 - zq^j)(q;\, q)_j (q;\, q)_{M-j}}.$$

Proof. Set

$$F(z) = \sum_{j=0}^{M} \frac{q^j}{(z;\, q)_{j+1}}.$$

Performing partial fraction decomposition on $F(z)$, we have

$$F(z) = \sum_{k=0}^{M} \frac{b_k}{1 - zq^k}. \tag{4.3}$$

For $0 \le k \le M$, we have

$$\begin{aligned}
b_k &= \lim_{z \to q^{-k}} (1 - zq^k) F(z) \\
&= \lim_{z \to q^{-k}} (1 - zq^k) \sum_{j=0}^{M} \frac{q^j}{(z;\, q)_{j+1}} \\
&= \lim_{z \to q^{-k}} (1 - zq^k) \sum_{j=k}^{M} \frac{q^j}{(z;\, q)_{j+1}} \\
&= \lim_{z \to q^{-k}} \sum_{j=k}^{M} \frac{q^j}{(z;\, q)_k (zq^{k+1};\, q)_{j-k}} \\
&= \sum_{j=k}^{M} \frac{q^j}{(q^{-k};\, q)_k (q;\, q)_{j-k}} \\
&= \frac{(-1)^k q^{\binom{k}{2}+2k}}{(q;\, q)_k} \sum_{j=0}^{M-k} \frac{q^j}{(q;\, q)_j} \\
&= \frac{(-1)^k q^{\binom{k}{2}+2k}}{(q;\, q)_k (q;\, q)_{M-k}}.
\end{aligned}$$

The last equation follows from the q-Chu-Vandermonde identity (4.2) by setting $a = 0$ and $c = q^{-n}$.

According to (4.3), we complete the proof of Theorem 4.2. □

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